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Notes on elation generalized quadrangles

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Abstract

Let $\mathcal{S}$ be a finite generalized quadrangle of order (s, t), $s, t > 1$. An "elation about a point p" of $\mathcal{S}$ is an automorphism fixing p linewise and fixing no point which is not collinear with p. An elation that generates a cyclic group of elations is called a "standard elation". One of the problems already considered in Payne and Thas (Finite Generalized Quadrangles (1984)) is to determine just when the set of elations about the point (∞) is a group. The purpose of this paper is to provide an example where this is not the case, and then to show that for a flock generalized quadrangle the usual group of elations about (∞) is the complete set of standard elations about (∞).

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1. Introduction

A (finite) *generalized quadrangle (GQ)* of order (s, t), $s \geq 1$ and $t \geq 1$ and $s, t \in \mathbb{N}$, is a point-line incidence structure $\mathcal{S} = (P, B, I)$ in which P and B are disjoint (non-empty) sets of objects called "points" and "lines" respectively, and for which I is a symmetric point-line incidence relation satisfying the following axioms.

(i) Each point is incident with $t + 1$ lines, and two distinct points are incident with at most one line.

(ii) Each line is incident with $s + 1$ points, and two distinct lines are incident with at most one point.

(iii) If p is a point and L is a line not incident with p, then there is a unique point–line pair (q, M) such that $pIMIqIL$.

If $s = t$, then $\mathcal{S}$ is also said to be "of order s".

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Let p and q be (not necessarily distinct) points of the GQ $\mathcal{S}$; we write $p \sim q$ and say that p and q are *collinear*, provided that there is some line L so that $pILIq$ (so $p \nsim q$ means that p and q are *not* collinear). Dually, for $L, M \in B$, we write $L \sim M$ or $L \nsim M$ accordingly as L and M are *concurrent* or *non-concurrent*. For $p \in P$, put $p^{\perp} = \{q \in P \parallel q \sim p\}$, and note that $p \in p^{\perp}$. For a pair of distinct points $\{p, q\}$, $\{p, q\}^{\perp}$ is defined as $p^{\perp} \cap q^{\perp}$. Then $|\{p, q\}^{\perp}| = s + 1$ or $t + 1$, accordingly as $p \sim q$ or $p \nsim q$. More generally, if $A \subseteq P$, $A^{\perp}$ is defined by $A^{\perp} = \bigcap\{p^{\perp} \parallel p \in A\}$. For $p \neq q$, $\{p, q\}^{\perp\perp} = \{r \in P \parallel r \in s^{\perp}$ for all $s \in \{p, q\}^{\perp}\}$. We have that $|\{p, q\}^{\perp\perp}| = s + 1$ or $|\{p, q\}^{\perp\perp}| \leq t + 1$, accordingly as $p \sim q$ or $p \nsim q$. If $p \sim q$, $p \neq q$, or if $p \nsim q$ and $|\{p, q\}^{\perp\perp}| = t + 1$, then we say that the pair $\{p, q\}$ is *regular*. The point p is *regular* provided $\{p, q\}$ is regular for every $q \in P \backslash \{p\}$.

A *subquadrangle*, or also *subGQ*, $\mathcal{S}' = (P', B', I')$ of a GQ $\mathcal{S} = (P, B, I)$ is a GQ for which $P' \subseteq P$, $B' \subseteq B$, and where I' is the restriction of I to $(P' \times B') \cup (B' \times P')$.

Let $\mathcal{S}$ be a finite generalized quadrangle with parameters (s, t), $s \neq 1 \neq t$, and let (∞) be a fixed point of $\mathcal{S}$. An *elation about the point* (∞) is a collineation of $\mathcal{S}$ that fixes each line through (∞) and fixes no point not collinear with (∞), or is the identity collineation. More generally, a *whorl about the point* (∞) is just a collineation fixing (∞) linewise. If there is a group K of s^2t elations acting regularly on the points of $\mathcal{S}$ not collinear with (∞), we say that $\mathcal{S}$ is an *elation generalized quadrangle (EGQ)* with *base point* (∞), or that $(\mathcal{S}^{(\infty)}, K)$ is an elation generalized quadrangle. If there is a group of whorls about (∞) acting transitively on the points not collinear with (∞), we say that (∞) is a *center of transitivity*.

If G is a group of order s^2t and $\mathcal{J}$ (respectively $\mathcal{J}^*$) is a set of $t + 1$ subgroups H_i (respectively H_i^*) of G of order s (respectively of order st), and if the conditions (K1) and (K2) below are satisfied, then the H_i^* are uniquely defined by the H_i, and $(\mathcal{J}, \mathcal{J}^*)$ is said to be a *4-gonal family of type* (s, t) in G.

(K1) $H_i H_j \cap H_k = \mathbf{1}$ for distinct i, j and k;
(K2) $H_i^* \cap H_j = \mathbf{1}$ for distinct i and j.

Theorem 1.1 (Kantor [4]). *From an EGQ $(\mathcal{S}^{(\infty)}, K)$ of order $(s, t), s, t > 1$, can be constructed a 4-gonal family of type (s, t) in K, and conversely, so that EGQ's of order (s, t) and 4-gonal families of type (s, t) are equivalent objects.*

One of the problems already considered in FGQ is to determine just when the set of elations about the point (∞) is a group. An elation that generates a cyclic group of elations is called a *standard elation*. The purpose of this paper is to provide an example where this is *not* the case, and then to show that for a flock generalized quadrangle (see below), the usual group of elations about (∞) is the complete set of standard elations about (∞).

2. q-Clans, flocks and flock GQ's

Let $\mathbb{F} = \mathbf{GF}(q), q$ any prime power, and put $K = \{(\alpha, c, \beta) \parallel \alpha, \beta \in \mathbb{F}^2, c \in \mathbb{F}\}$. Define a binary operation on K by

$$(\alpha, c, \beta) \circ (\alpha', c', \beta') = (\alpha + \alpha', c + c' + \beta \circ \alpha', \beta + \beta')$$

where $\gamma \circ \gamma'$ denotes $\gamma \gamma'^{\mathrm{T}}$.

This makes K into a group whose center is $\mathfrak{C} = \{(\bar{0}, c, \bar{0}) \in K \parallel c \in \mathbb{F}\}$.
Let $\mathcal{C} = \{A_u \parallel u \in \mathbb{F}\}$ be a set of q distinct upper triangular 2×2-matrices over $\mathbb{F}$. Then $\mathcal{C}$ is called a *q-clan* provided $A_u - A_r$ is *anisotropic* whenever $u \neq r$, i.e. $\alpha(A_u - A_r)\alpha^{\mathrm{T}} = 0$ has only the trivial solution $\alpha = (0, 0)$. For $A_u \in \mathcal{C}$, put $K_u = A_u + A_u^{\mathrm{T}}$. Let

$$A_u = \begin{pmatrix} x_u & y_u \\ 0 & z_u \end{pmatrix}, \qquad x_u, y_u, z_u, u \in \mathbb{F}.$$

For q odd, $\mathcal{C}$ is a q-clan if and only if

$$-\det(K_u - K_r) = (y_u - y_r)^2 - 4(x_u - x_r)(z_u - z_r) \tag{1}$$

is a non-square of $\mathbb{F}$ whenever $r, u \in \mathbb{F}, r \neq u$. For q even, $\mathcal{C}$ is a q-clan if and only if

$$y_u \neq y_r \qquad \text{and} \qquad \mathrm{tr}((x_u + x_r)(z_u + z_r)(y_u + y_r)^{-2}) = \mathbf{1} \tag{2}$$

whenever $r, u \in \mathbb{F}, r \neq u$.

Now we can define a family of subgroups of K by

$$A(u) = \{(\alpha, \alpha A_u \alpha^{\mathrm{T}}, \alpha K_u) \in K \parallel \alpha \in \mathbb{F}^2\}, \qquad u \in \mathbb{F},$$

and

$$A(\infty) = \{(\bar{0}, 0, \beta) \in K \parallel \beta \in \mathbb{F}^2\}.$$

Then put $\mathcal{J} = \{A(u) \parallel u \in \mathbb{F} \cup \{\infty\}\}$ and $\mathcal{J}^* = \{A^*(u) \parallel u \in \mathbb{F} \cup \{\infty\}\}$, with $A^*(u) = A(u)\mathfrak{C}$. So

$$A^*(u) = \{(\alpha, c, \alpha K_u) \in K \parallel \alpha \in \mathbb{F}^2, c \in \mathbb{F}\}, \qquad u \in \mathbb{F},$$

and

$$A^*(\infty) = \{(\bar{0}, c, \beta) \parallel \beta \in \mathbb{F}^2, c \in \mathbb{F}\}.$$

With $K, A(u), A^*(u), \mathcal{J}$ and $\mathcal{J}^*$ as above, the following important theorem is a combination of results of Payne [6, 7] and Kantor [4].

Theorem 2.1 ([4, 6, 7]). *The pair $(\mathcal{J}, \mathcal{J}^*)$ is a 4-gonal family for K if and only if $\mathcal{C}$ is a q-clan. Hence if $\mathcal{C}$ is a q-clan, then it defines a GQ $\mathcal{S}(\mathcal{C})$ of order (q^2, q).*

The GQ $\mathcal{S}(\mathcal{C})$ of order (q^2, q) is constructed as follows.

- The POINTS are of three types:
  (1) the elements $g = (\alpha, c, \beta)$ of K;
  (2) cosets $A^*(t)g, t \in \mathbb{F} \cup \{\infty\}, g \in K$;
  (3) the symbol (∞).
- The LINES are of two types:
  (1) cosets $A(t)g, t \in \mathbb{F} \cup \{\infty\}, g \in K$;
  (2) symbols $[A(t)], t \in \mathbb{F} \cup \{\infty\}$.
- INCIDENCE. The point (∞) is on the $q+1$ lines $[A(t)]$ of Type (2). The point $A^*(t)g$ is on the line $[A(t)]$ and on the q lines of Type (1) contained in $A^*(t)g$. The point g of Type (1) is on the $q + 1$ lines $A(t)g$ of Type (1) that contain it. There are no further incidences.

For each $g = (\alpha, c, \beta) \in K$, right multiplication by g induces an elation about (∞) denoted by $\pi(g) = \pi(\alpha, c, \beta)$. Then if $\hat{K}$ denotes the set $\{\pi(g) \parallel g \in K\}$, $\hat{K}$ is a group of s^2t elations about (∞) acting regularly on the points not collinear with (∞) and $\pi : K \to \hat{K} : g \mapsto \pi(g)$ is an isomorphism.

Remark 2.2. With $\mathcal{S}(\mathcal{C})$ as above, it is easy to see that (∞) is a regular point, because the center $\mathfrak{C}$ is a maximal group of symmetries about (∞) (recall that a *symmetry* about a point is an automorphism fixing every point collinear with it). Also, the reader should keep in mind that

$$\{(\infty), (\bar{0}, 0, \bar{0})\}^{\perp\perp} = \{(\bar{0}, c, \bar{0}) \parallel c \in \mathbb{F}\} \cup \{(\infty)\}.$$

Let $\mathcal{F}$ be a *flock* of the quadratic cone $\mathcal{K}$ with vertex v of $\mathbf{PG}(3, q)$, that is, a partition of $\mathcal{K} \backslash \{v\}$ into q disjoint (irreducible) conics. In his paper on flock geometry [11], Thas showed in an algebraic way that (1) and (2) are exactly the conditions for the planes

$$x_u X_0 + z_u X_1 + y_u X_2 + X_3 = 0$$

of $\mathbf{PG}(3, q)$ to define a flock of the quadratic cone $\mathcal{K}$ with equation $X_0 X_1 = X_2^2$. Hence we have the following theorem.

Theorem 2.3 (Thas [11]). *To any flock $\mathcal{F}$ of the quadratic cone of* $\mathbf{PG}(3, q)$ *corresponds a GQ $\mathcal{S}$ of order (q^2, q).*

We denote by $\mathcal{S}(\mathcal{F})$ the GQ of order (q^2, q) which arises from $\mathcal{F}$ as in Theorem 2.3, and such a GQ is called a *flock generalized quadrangle*.

Now define the following q-clan $\mathcal{C}$:

$$\mathcal{C} = \left\{ A_t = \begin{pmatrix} t & 0 \\ 0 & -mt^{\sigma} \end{pmatrix} \middle\| \, t \in \mathbb{F} \right\},$$

where q is odd, m a given non-square and $\sigma \in \mathrm{Aut}(\mathbb{F})$.

So $\mathcal{S}$ is the Kantor–Knuth semifield flock GQ.

Let $Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, so $Q = Q^{\mathrm{T}} = Q^{-1}$. It is noted in [8] that the map

$$\tau : (\alpha, c, \beta) \mapsto (\alpha Q, c, \beta Q)$$

is an automorphism of K which induces a collineation of $\mathcal{S}$ that fixes $(\bar{0}, 0, \bar{0})$ and (∞) linewise. Let $g = (\alpha', c', \beta')$ be a fixed element of K to be determined later. Put

$$\theta = \tau \circ \pi(g) = \tau \circ \pi(\alpha', c', \beta').$$

For the appropriate choice of g we will establish the following:

(1) θ is an elation about (∞) that is not in $\hat{K}$;

(2) θ^p, where $q = p^h$ for some h and the odd prime p, is an involution whose fixed element structure is a subquadrangle of order q, so that in particular θ^p is not an elation about (∞);

(3) $\theta^2 \in \hat{K}$.

The fact that some power of θ (i.e. θ^p) is not an elation about (∞) suggests that possibly we have been using an unsatisfactory definition of elation. Let us say that a *standard elation* about (∞) is an elation about (∞) that acts semiregularly on the points not collinear with (∞), i.e. a standard elation about (∞) is a collineation of $\mathcal{S}$ that generates a cyclic group of elations about (∞). In Section 5, we will then obtain the following:

Theorem 2.4. *If $\mathcal{S}$ is a flock GQ with the usual notation, the set of standard elations about the point (∞) is a group. Whence the usual elation group $\hat{K}$ is the complete set of standard elations about (∞).*

3. The nitty gritty

Let $\mathcal{S}$ be the Kantor–Knuth semifield flock GQ given in the previous section, and let Q be as defined there.

Lemma 3.1. *If j is a positive integer, let j_2 denote the element of $\{0, 1\}$ to which j is congruent modulo 2. Similarly, let $-j_2$ denote -1 if j is odd and 0 if j is even. Also, I is the 2×2 identity matrix.*

$$I + Q + Q^2 + \cdots + Q^j = \begin{pmatrix} j+1 & 0 \\ 0 & (j+1)_2 \end{pmatrix}. \tag{3}$$

$$Q + Q^2 + \cdots + Q^j = \begin{pmatrix} j & 0 \\ 0 & -j_2 \end{pmatrix}. \tag{4}$$

$$Q^{j-1} + 2Q^{j-2} + 3Q^{j-3} + \cdots + (j-1)Q^1 = \begin{pmatrix} \frac{(j-1)j}{2} & 0 \\ 0 & -\lfloor \frac{j}{2} \rfloor \end{pmatrix}. \tag{5}$$

In Eqs. (3) *and* (4) *we take $j \geq 1$. In Eq.* (5) *we must take $j \geq 2$.*

Proof. All three equations are established by routine induction arguments (note that $\lfloor \frac{j}{2} \rfloor = \frac{j-j_2}{2}$). $\square$

By definition we have

$$\theta = \tau \circ \pi(\alpha', c', \beta') : (\alpha, c, \beta) \mapsto (\alpha Q + \alpha', c + c' + \beta Q \circ \alpha', \beta Q + \beta'). \tag{6}$$

Then by an easy calculation we have

$$\begin{aligned} \theta^2 : (\alpha, c, \beta) \mapsto (&\alpha Q^2 + \alpha'(Q + I), c + 2c' + \beta(Q + Q^2) \circ \alpha' \\ &+ \beta' Q \circ \alpha', \beta Q^2 + \beta'(Q + I)). \end{aligned} \tag{7}$$

It now follows by a routine induction that

$$\begin{aligned} \theta^i : (\alpha, c, \beta) \mapsto (&\alpha Q^i + \alpha'(Q^{i-1} + Q^{i-2} + \cdots + Q + I), c + ic' + \beta(Q + Q^2 \\ &+ Q^3 + \cdots + Q^i) \circ \alpha' + \beta'(Q^{i-1} + 2Q^{i-2} + 3Q^{i-3} + \cdots \\ &+ (i-1)Q^1) \circ \alpha', \beta Q^i + \beta'(Q^{i-1} + Q^{i-2} + \cdots + Q + I)) \end{aligned}$$

$$= \left(\alpha\begin{pmatrix}1 & 0\\ 0 & (-1)^i\end{pmatrix} + \alpha'\begin{pmatrix}i & 0\\ 0 & i_2\end{pmatrix},\ c + ic' + \beta\begin{pmatrix}i & 0\\ 0 & -i_2\end{pmatrix}\circ\alpha'\right.$$
$$\left. + \beta'\begin{pmatrix}\frac{(i-1)i}{2} & 0\\ 0 & -\lfloor\frac{i}{2}\rfloor\end{pmatrix}\circ\alpha', \beta\begin{pmatrix}1 & 0\\ 0 & (-1)^i\end{pmatrix} + \beta'\begin{pmatrix}i & 0\\ 0 & i_2\end{pmatrix}\right). \quad (8)$$

At this stage it is convenient to have written out the image of θ^i separately for odd and even i.

$$\theta^{2k} : (\alpha, c, \beta) \mapsto \left(\alpha + \alpha'\begin{pmatrix}2k & 0\\ 0 & 0\end{pmatrix}, c + 2kc' + \beta\begin{pmatrix}2k & 0\\ 0 & 0\end{pmatrix}\circ\alpha'\right.$$
$$\left. + \beta'\begin{pmatrix}(2k-1)k & 0\\ 0 & -k\end{pmatrix}\circ\alpha', \beta + \beta'\begin{pmatrix}2k & 0\\ 0 & 0\end{pmatrix}\right). \quad (9)$$

$$\theta^{2k+1} : (\alpha, c, \beta) \mapsto \left(\alpha\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \alpha'\begin{pmatrix}2k+1 & 0\\ 0 & 1\end{pmatrix}, c + (2k+1)c'\right.$$
$$+ \beta\begin{pmatrix}2k+1 & 0\\ 0 & -1\end{pmatrix}\circ\alpha' + \beta'\begin{pmatrix}k(2k+1) & 0\\ 0 & -k\end{pmatrix}\circ\alpha',$$
$$\left.\beta\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} + \beta'\begin{pmatrix}2k+1 & 0\\ 0 & 1\end{pmatrix}\right). \quad (10)$$

We now determine whether or not θ^i fixes some point (α, c, β). First, θ^{2k} fixes (α, c, β) if and only if

(i) $\alpha'\begin{pmatrix}2k & 0\\ 0 & 0\end{pmatrix} = (0, 0)$;

(ii) $2kc' + \beta\begin{pmatrix}2k & 0\\ 0 & 0\end{pmatrix}\circ\alpha' + \beta'\begin{pmatrix}(2k-1)k & 0\\ 0 & -k\end{pmatrix}\circ\alpha' = 0$;

and

(iii) $\beta'\begin{pmatrix}2k & 0\\ 0 & 0\end{pmatrix} = (0, 0)$.

It follows readily that if $\alpha' = (a_1, a_2)$ with $a_1 \neq 0$, then we have proved the following lemma:

Lemma 3.2. *Assume that $a_1 \neq 0$. Then θ^{2k} fixes some (α, c, β) if and only if $k \equiv 0 \bmod p$, in which case $\theta^{2k} = id$. In particular, θ^2 is an elation, even a standard elation.*

Note that it is also easy to check that $\theta^2 \in \hat{K}$.

Now we consider whether or not θ^{2k+1} fixes some (α, c, β). Since we are assuming that $\alpha' = (a_1, a_2)$ with $a_1 \neq 0$, it follows easily that if θ^{2k+1} fixes some (α, c, β) it must be the case that $\alpha\begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix} = \alpha'\begin{pmatrix}2k+1 & 0\\ 0 & 1\end{pmatrix}$, which forces $2k + 1 \equiv 0 \bmod p$. So consider the fixed points of θ^p. It is routine to check that θ^p fixes the point (α, c, β) if and only if $(\alpha, c, \beta) = ((a, -ka_2), c, (b, -kb_2))$, $a, c, b \in \mathbb{F}$, where $\alpha' = (a_1, a_2)$, $\beta' = (b_1, b_2)$ and $p = 2k + 1$.

It now follows that θ^p is an involution with a subquadrangle of order q as its fixed element structure.

4. The kernel of a flock GQ

Suppose $\mathcal{S}(\mathcal{F}) = (P, B, I)$ is a flock GQ of order (q^2, q) which arises from the flock $\mathcal{F}$ of the quadratic cone $\mathcal{K}$ in $\mathbf{PG}(3, q)$, and suppose $\mathcal{C} = \{A_t \parallel t \in \mathbb{F}\}$ is the associated normalized q-clan [9] (i.e. the A_i are symmetric if q is odd, uppertriangular if q is even, and A_0 is the zero matrix). Suppose (∞) is the special point of $\mathcal{S}(\mathcal{F})$. Then consider $[A(\infty)]I(\infty)$ and the point $(\bar{0}, 0, \bar{0})$ of $P\backslash(\infty)^\perp$. Let θ be a collineation of $\mathcal{S}(\mathcal{F})$ which fixes (∞), $[A(\infty)]$ and $(\bar{0}, 0, \bar{0})$. Then by Payne [9], the following must exist:

(i) a permutation $\pi : t \mapsto \bar{t}$ of the elements of $\mathbb{F} = \mathbf{GF}(q)$;
(ii) $\tau \in \mathrm{Aut}(\mathbf{GF}(q))$;
(iii) $\lambda \in \mathbf{GF}(q)$, $\lambda \neq 0$;
(iv) $D \in \mathbf{GL}(2, q)$ for which $A_{\bar{t}} - \lambda D^{\mathrm{T}}(A_t - A_0)^\tau D - A_{\bar{0}}$ is skew-symmetric (with zero diagonal) for all $t \in \mathbf{GF}(q)$.

Conversely, given τ, D, λ and the permutation $\pi : x \mapsto \bar{x}$ satisfying condition (iv), a collineation $\theta = \theta(\tau, D, \lambda, \pi)$ of the GQ $\mathcal{S}(\mathcal{F})$ arises, as follows:

$$\begin{aligned}\theta = \theta(\tau, D, \lambda, \pi) : (\alpha, c, \beta) \mapsto (&\lambda^{-1}\alpha^\tau D^{-T}, \lambda^{-1}(c - \alpha A_0 \alpha^{\mathrm{T}})\\ &+ \lambda^{-2}\alpha^\tau(D^{-T} A_{\bar{0}} D^{-1})(\alpha^\tau)^{\mathrm{T}}, (\beta - \alpha K_0)^\tau D + \lambda^{-1}\alpha^\tau D^{-T} K_{\bar{0}}).\end{aligned}$$

For $\theta = \theta(\tau, D, \lambda, \pi)$, write $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Define a projective semilinear collineation T_θ of $\mathbf{PG}(3, q)$ as follows (defined on the planes of $\mathbf{PG}(3, q)$):

$$T_\theta : \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \mapsto \begin{pmatrix} \lambda a^2 & \lambda ab & \lambda b^2 & x_{\bar{0}} \\ 2\lambda ac & \lambda(ad + bc) & 2\lambda bd & y_{\bar{0}} \\ \lambda c^2 & \lambda cd & \lambda d^2 & z_{\bar{0}} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} x^\tau \\ y^\tau \\ z^\tau \\ 1 \end{bmatrix}.$$

Then T_θ fixes the cone $\mathcal{K}$ and leaves invariant $\mathcal{F}$ precisely when θ is a collineation of $\mathcal{S}(\mathcal{F})$. The map $T : \theta \mapsto T_\theta$ is a homomorphism from the subgroup of $\mathrm{Aut}(\mathcal{S}(\mathcal{F}))$ leaving (∞), $[A(\infty)]$ and $(\bar{0}, 0, \bar{0})$ invariant onto the subgroup of $\mathbf{P\Gamma O}(4, q)$ leaving the flock $\mathcal{F}$ invariant. The kernel $N(T)$ of T is

$$N(T) = \{\theta_a \parallel (\alpha, c, \beta) \mapsto (a\alpha, a^2 c, a\beta), 0 \neq a \in \mathbf{GF}(q)\}.$$

Note that $N(T)$ is a group of generalized homologies with centers (∞) and $(\bar{0}, 0, \bar{0})$ (i.e. fixing (∞) and $(\bar{0}, 0, \bar{0})$ linewise).

The following observation is taken from Payne [9]:

Lemma 4.1 (Payne [9]). *Suppose $\mathcal{S}(\mathcal{F})$ is a non-classical flock GQ of order (q^2, q), $q > 1$, and let H be the full group of automorphisms of $\mathcal{S}(\mathcal{F})$ fixing (∞) and $(\bar{0}, 0, \bar{0})$ linewise.*

(i) *If $q = 2^h$, then $H = N(T)$;*
(ii) *If q is odd, and $H^* = \{\theta(\tau, D, \lambda, \pi) \in H \parallel \tau = \mathbf{1}\}$, then $H^* = N(T)$, except if $\mathcal{F}$ is a Kantor–Knuth semifield flock. In that case, $H = H^*$ if $\sigma^2 \neq \mathbf{1}$, and then $[H : N(T)] = 2$. If $\sigma^2 = \mathbf{1} \neq \sigma$, then $[H : H^*] = [H^* : N(T)] = 2$.*

In view of Lemma 4.1 and of the rest of this paper, we point out the existence of the following rather remarkable collineation ϕ^- if q is odd (for all flocks!). Suppose q is odd, and put $\phi^- = \theta_{-1}$. So

$$\phi^- : (\alpha, c, \beta) \mapsto (-\alpha, c, -\beta).$$

Then ϕ^- is an involution of $\mathcal{S}(\mathcal{F})$ which fixes (∞) and $(\bar{0}, 0, \bar{0})$ linewise, and which fixes each point of $\{(\infty), (\bar{0}, 0, \bar{0})\}^{\perp\perp} = \{(\bar{0}, c, \bar{0}) \parallel c \in \mathbb{F}\} \cup \{(\infty)\}$. We remark that in the Kantor–Knuth case (or more generally, in the case of the translation duals of the semifield flock GQ's in odd characteristic), the existence of this collineation was pointed out in Thas [15, 16] without coordinates.

Now suppose ϕ is an automorphism of the flock GQ $\mathcal{S}(\mathcal{F})$ of order (q^2, q), $q(>1)$ odd, and suppose ϕ fixes (∞) and $(\bar{0}, 0, \bar{0})$ linewise, and each point of $\{(\infty), (\bar{0}, 0, \bar{0})\}^{\perp\perp} = \{(\bar{0}, c, \bar{0}) \parallel c \in \mathbb{F}\} \cup \{(\infty)\}$. We can write ϕ as $\phi(\tau, D, \lambda, \pi)$. As each point of the form $(\bar{0}, c, \bar{0})$ is fixed by ϕ, the explicit form of $\phi(\tau, D, \lambda, \pi)$ above leads to the fact that $\tau = \mathbf{1}$ (and $\lambda = 1$). Hence by Lemma 4.1, ϕ is an element of $N(T)$ if $\mathcal{F}$ is not Kantor–Knuth. It follows immediately that

$$\phi = \phi^- : (\alpha, c, \beta) \mapsto (-\alpha, c, -\beta),$$

and thus ϕ is an involution.

5. Standard elations in flock GQ's

We start with a general lemma.

We first recall a notation. Suppose $p \not{I} L$. Then by $proj_L p$, we denote the unique point on L collinear with p. Dually, $proj_p L$ is the unique line incident with p concurrent with L.

Lemma 5.1. *Suppose $\mathcal{S} = (P, B, I)$ is a GQ of order (s, t), $s, t > 1$, with s and t powers of the same prime number p. Suppose (∞) is a regular point which is a center of transitivity, and let H be the full group of whorls about (∞). Let G be a Sylow p-subgroup of H. Then we either have*

(i) $|G| = s^2 t$, *or*
(ii) $p = 2$, $|G| = 2s^2 t$, *and $\mathcal{S}$ contains a (proper) subGQ of order t isomorphic to $W(t)$; consequently, $s = t^2$.*

Proof. As H acts transitively on $P \backslash \{(\infty)\}^{\perp}$, $s^2 t$ is a divisor of $|H|$.

Suppose $s^2 t$ is not the largest power of p dividing $|H|$. Consider a point $x \not\sim (\infty)$. Then H_x contains a non-trivial element of order p, say θ. As s is a power of p, θ fixes a point of $L \backslash \{(\infty), proj_L x\}$, where L is an arbitrary line incident with (∞). By Thas [14], the fixed element structure of θ is a subGQ $\mathcal{S}'$ of order t, and $s = t^2$. Also, θ necessarily is an involution, hence $p = 2$ in that case. By Thas [12], it follows that $\mathcal{S}' \cong W(t)$. It also follows easily that $|G| = 2s^2 t$. $\square$

From now on, we suppose that $\mathcal{S}(\mathcal{F})$ is a flock GQ of order (q^2, q), $q > 1$ and $q = p^h$ for the prime p, $\mathcal{F}$ not a linear flock. By K (which we identify with $\hat{K}$ for convenience), we denote the elation group as defined in Section 2, and by $\mathfrak{C}$, its center (as indentified in the same section), which is a maximal group of symmetries of size q about (∞). Also, H denotes the full group of whorls about (∞).

Lemma 5.2. *Let $\{G_0 = K, G_1, \dots, G_r\}$ be the set of all maximal elation groups with elation point (∞). Then each G_i is a Sylow p-subgroup, thus all maximal elation groups with elation point (∞) are conjugate.*

Proof. For q odd, the lemma follows from Lemma 5.1. Let q be even. As $\mathcal{F}$ is not linear, by O'Keefe and Penttila [5] $\mathcal{S}(\mathcal{F})$ contains no classical subGQ's of order q containing (∞). So the lemma follows from Lemma 5.1. □

Corollary 5.3. *Assume that $\mathcal{S}(\mathcal{F})$ is a flock GQ of order (q^2, q), $q > 1$ and $q = p^h$ for the prime p, $\mathcal{F}$ not a linear flock. Then each whorl about (∞) of order p^k for $k \neq 0$ is a standard elation, and conversely.*

Proof. Immediate. □

Corollary 5.4. *$\mathfrak{C}$ is the center of each of the elation groups G_i, $i = 0, 1, \dots, r$.*

Proof. Immediately from the fact that the G_i are conjugate, and that $\mathfrak{C}$ is in the center of $K = G_0$. □

Now we introduce *Property* $(M)_{(\infty)}$ w.r.t. G_j, $j \in \{0, 1, \dots, r\}$:

PROPERTY $(M)_{(\infty)}$. *Let M be an arbitrary line not incident with (∞). Put $m = proj_M(\infty)$. If $\theta \in G_j$ fixes M, then θ fixes each line incident with m.*

Lemma 5.5. *For each $j \in \{0, 1, \dots, r\}$, Property $(M)_{(\infty)}$ holds for G_j.*

Proof. Suppose M and m are as above, and let $\mathbf{1} \neq \theta \in G_j$ fix M. Suppose $N I m$ is a line for which $N^\theta \neq N$. Let $\phi \in \mathfrak{C}$ be so that $M^\phi = N$. Then $M^{\theta\phi} = N$, while $M^{\phi\theta} = N^\theta \neq N$, contradicting the fact that $\mathfrak{C}$ is the center of G_j. □

Lemma 5.6. *Let $y \in (\infty)^\perp \setminus \{(\infty)\}$. Suppose H_y^* is the subgroup of H which fixes y linewise. Then H_y^* has a unique (Sylow p-) subgroup $H(y)$ of size q^2.*

Proof. By Lemma 5.5, it is clear that $|H_y^*|_p \geq q^2$, where $|H_y^*|_p$ is the largest power of p dividing $|H_y^*|$.

Suppose equality does not hold. Then there is some non-trivial element of H_y^* fixing some point $x \sim y, x \nsim (\infty)$, whence fixing the dual grid defined by (∞) and x elementwise.

Suppose q is even. Then this contradicts Lemma 4.1.

Now suppose that q is odd and that $\mathcal{F}$ is not Kantor–Knuth. Then by the end of the previous section, it follows immediately that $|H_y^*| = 2q^2$. Suppose $H(y)$ and $H(y)'$ are two distinct Sylow p-subgroups of H_y^*, $q = p^h$ for the odd prime p. Then clearly $|H_y^*| \geq pq^2$, contradicting the fact that $|H_y^*| = 2q^2$, as p is odd.

Finally, suppose that $\mathcal{F}$ is a Kantor–Knuth semifield flock. As the point (∞) is a regular point, the incidence structure $\mathcal{N}^D$ with point set $(\infty)^\perp \setminus \{(\infty)\}$, with line set the set of spans $\{q, r\}^{\perp\perp}$, where q and r are non-collinear points of $(\infty)^\perp \setminus \{(\infty)\}$, and with the natural incidence, is the dual of a net of order q^2 and degree $q + 1$ [10, 1.3.1]. As $\mathcal{F}$ is a

Kantor–Knuth semifield flock, by Thas and Van Maldeghem [13] $\mathcal{N}^D$ is the dual net H_q^3, which is constructed as follows:

- the POINTS of H_q^3 are the points of $\mathbf{PG}(3, q)$ not on a given line U of $\mathbf{PG}(3, q)$;
- the LINES of H_q^3 are the lines of $\mathbf{PG}(3, q)$ which have no point in common with U;
- the INCIDENCE in H_q^3 is the natural one.

Suppose there are distinct groups $H(y)^1$ and $H(y)^2$ which have size q^2, both fixing (∞) and y linewise. Interpreted in $\mathcal{N}^D \cong H_q^3$, the $H(y)^i$ faithfully induce collineation groups of $\mathbf{PG}(3, q)$ fixing U and a point $y' \not\mathrm{I}\, U$ (corresponding to y). Also, the $q + 1$ planes through U (corresponding to the lines through (∞)) are fixed by both groups. Thus they are subgroups of $\mathbf{PGL}(4, q)_U$. For each $H(y)^i$, there is at least one other line $Y_i \mathrm{I} y'$ in the plane $\langle y', U\rangle$ which is fixed by $H(y)^i$. As each $\theta_i \in H(y)^i$ has order p^r for some r, each such θ_i also fixes some point z_i on Y_i which is different from y' and $Y_i \cap U$. As θ_i is an element of $\mathbf{PGL}(4, q)$, it follows that θ_i (and hence each element of $H(y)^i$) fixes each point on Y_i. So $H(y)^i$ fixes each line through $Y_i \cap U$ (note that $H(y)^i$ does not act faithfully on $\langle y', U\rangle$).

Now suppose we cannot choose $Y_1 = Y_2$, namely Y_i is unique for $i = 1, 2$ and $Y_1 \neq Y_2$. Then $\langle H(y)^1, H(y)^2\rangle$ induces a collineation group of size divisible by $(q + 1)q(q - 1)$ on the plane $\langle y', U\rangle$ (as that plane is Desarguesian). This clearly contradicts the end of the previous section when $\langle H(y)^1, H(y)^2\rangle$ is interpreted as a collineation group of $\mathcal{S}(\mathcal{F})$.

Whence $Y_1 = Y_2$, and both $H(y)^1$ and $H(y)^2$, as collineation groups of $\mathcal{S}(\mathcal{F})$, fix the same $q + 1$ points on $y(\infty)$. We keep working in $\mathcal{S}(\mathcal{F})$. As $H(y)^1 \neq H(y)^2$, $\langle H(y)^1, H(y)^2\rangle$ has a non-trivial element ϕ which fixes some point $x \sim y, x \not\sim (\infty)$. It immediately follows that ϕ fixes a subGQ of $\mathcal{S}$ of order q pointwise, and ϕ is an involution (see, e.g., [14]). So

$$|\langle H(y)^1, H(y)^2\rangle| = 2q^2.$$

But as before, this contradicts the fact that q is odd. The lemma follows. □

Note. It is clear that $H(y)$ acts regularly on the q^2 points different from y of an arbitrary line $N \mathrm{I} y$, $N \neq (\infty)y$.

Remark 5.7. Suppose $\mathcal{S}(\mathcal{F})$ is as in the preceding lemma, and suppose q is even. Suppose $H(y)^1 \neq H(y)^2$. By Thas and Van Maldeghem [13], $\mathcal{N}^D \cong H_q^3$. So taking over the proof of the second part of Lemma 5.6, $\mathrm{Aut}(\mathcal{S}(\mathcal{F}))$ contains a non-trivial involution fixing some subGQ $\mathcal{S}'$ of order q pointwise. By [12], $\mathcal{S}' \cong W(q)$, and by [5], $\mathcal{F}$ is linear.

6. Proof of Theorem 2.4

Theorem 6.1. *Let $\mathcal{S}(\mathcal{F})$ be a flock GQ of order (q^2, q), $q > 1$ and $q = p^h$ for the prime p. Then the set of standard elations about the point (∞) is a group. Whence the usual elation group $\hat{K} = K$ is the complete set of standard elations about (∞).*

Proof. Suppose that $\mathcal{F}$ is not linear (in particular, $q > 2$). It is clear that, with the notation of the previous section, it suffices to prove that $G_0 = G_1 = \cdots = G_r$. Fix a $j \in \{0, 1, \ldots, r\}$.

For each point $y \sim (\infty) \neq y$, by Lemma 5.5, G_j has a subgroup $G_j(y)$ of size q^2 each element of which fixes y linewise (the stabilizer $(G_j)_M$ of any line MIy in G_j, $M \neq (\infty)y$, has size q^2). But by Lemma 5.6, this group $G_j(y)$ is unique in H, and thus $G_0(y) = G_1(y) = \cdots = G_r(y)$ for each such y.

Now define for each $i = 0, 1, \ldots, r$ the group

$$G'_i = \langle G_i(y) \parallel y \sim (\infty) \neq y \rangle.$$

It is clear to the reader that if G'_i coincides with G_i for each i, then $G_0 = G_1 = \cdots = G_r$.

Suppose x and x' are arbitrary but different points not collinear with (∞). Then by a result of Brouwer [1], there are points $x = x_0, x_1, \ldots, x_n = x'$, all not collinear with (∞), so that $x_j \sim x_{j+1} \neq x_j$ for $0 \leq j < n$. Fix $i \in \{0, 1, \ldots, r\}$. For each $j = 0, 1, \ldots, n-1$, define y_j as the point of $(\infty)^\perp$ incident with $x_j x_{j+1}$. Then clearly there is a $\theta_j \in G_i(y_j)$ mapping x_j to x_{j+1}. Whence

$$x^{\theta_0\theta_1\ldots\theta_{n-1}} = x',$$

and G'_i acts transitively on the points not collinear with (∞). It follows that $G_i = G'_i$ for each $i = 0, 1, \ldots, r$, and hence that $G_0 = G_1 = \cdots = G_r$.

If $\mathcal{F}$ is linear, then the theorem is well-known. $\square$

7. Remarks on the general case

In this final section, we present some generalizations of results obtained previously. The proofs are related but slightly different—they all use the Theorem of Frobenius.

By the previous observations, we can say the following about the case $s = t$:

Theorem 7.1. *Suppose $\mathcal{S} = (P, B, I)$ is a GQ of order s, $s > 1$. Suppose (∞) is a regular point which is a center of transitivity. Then the set of all standard elations about (∞) is a group.*

Proof. Let H be the full group of whorls about (∞). Suppose $\mathbf{T}$ is the set of spans of non-collinear points in $(\infty)^\perp$. Then from Thas [14] follows that $(H/\mathfrak{C}, \mathbf{T})$ is a Frobenius group, where $\mathfrak{C} \trianglelefteq H$ is the group of symmetries about (∞) (as (∞) is a regular point which is a center of transitivity, by Thas [14] the group $\mathfrak{C}$ of symmetries about (∞) is a maximal group of symmetries (of size s) about (∞)). Let $F/\mathfrak{C}$ be the Frobenius kernel of $H/\mathfrak{C}$; then $F \leq H$ is a group of size s^3 which acts semiregularly, and hence regularly, on the points not collinear with (∞). Whence $\mathcal{S}$ is an EGQ, and by Frohardt [2], s is a prime power (say of the prime p). Let G be a Sylow p-subgroup of H. Then from the proof of Lemma 5.1 follows that $|G| = s^3$, and each element of G is a standard elation. Since F is a normal subgroup of H, we have that $G = F$ is the only Sylow p-subgroup in H. The theorem follows. $\square$

Let us call an EGQ $(\mathcal{S}^{(\infty)}, G)$ of order $(s, t), s, t > 1$, for which G contains a maximal group of symmetries (of size t) about (∞) a *skew translation generalized quadrangle (STGQ)*. For $s = t$, the latter condition is equivalent to saying that (∞) is a regular point, see [14]. We then have

Theorem 7.2. *Let $(\mathcal{S}^{(\infty)}, G)$ be an STGQ of order $s > 1$. Then the set of all standard elations about (∞) is a group.*

In fact, by applying the main result of Hachenberger [3], we can obtain the following more general result:

Theorem 7.3. *Let $(\mathcal{S}^{(\infty)}, G)$ be an STGQ of order (s, t), $s, t > 1$. Then we have one of the following possibilities:*

(i) *the set of all standard elations about (∞) is a group;*
(ii) *$s = t^2$, s is a power of 2, and there is a subGQ of order t isomorphic to $W(t)$ which contains the point (∞).*

Proof. By [3], s and t are powers of the same prime. Suppose that $\mathcal{S}$ has no subGQ of order (s', t), $s' > 1$, containing (∞), so that in particular we are not in case (ii). Then with H, $\mathbf{T}$ and $\mathfrak{C}$ as in the proof of Theorem 7.1, $(H/\mathfrak{C}, \mathbf{T})$ is a Frobenius group by Thas [14]. The theorem now follows in the same way as Theorem 7.1. □

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